

# A GEOMETRICAL CORRESPONDENCE BETWEEN MAXIMAL SURFACES IN ANTI-DE SITTER SPACE-TIME AND MINIMAL SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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**ABSTRACT.** A geometrical correspondence between maximal surfaces in anti-De Sitter space-time and minimal surfaces in the Riemannian product of the hyperbolic plane and the real line is established. New examples of maximal surfaces in anti-De Sitter space-time are obtained in order to illustrate this correspondence.

## 1. INTRODUCTION

The study of minimal surfaces in product spaces  $M^2 \times \mathbb{R}$  was initiated by Rosenberg and Meeks [Ros02, MR05] and has been very active since then. Among that spaces, there are three homogeneous Riemannian manifolds:  $\mathbb{R}^3$ , where the classical theory of minimal surfaces has been developed, and  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$ , where many authors have been actively working. Giving a complete list of references in the subject is far from being possible so we will only mention a few of them: Nelli and Rosenberg [NR02] proved a Jenkins-Serrin-type theorem in  $\mathbb{H}^2 \times \mathbb{R}$ , Hauswirth [Hau06] constructed minimal examples of Riemann type, Sá Earp and Tobiána [ST04] investigated the screw motion invariant surfaces in  $\mathbb{H}^2 \times \mathbb{R}$ , Daniel [Dan09] and Hauswirth, Sá Earp and Tobiána [HST08] showed, independently, the existence of an associated family of minimal immersions for simply connected minimal surfaces in  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$ , Urbano and the author [TU13] tackled a general study of minimal surfaces in  $\mathbb{S}^2 \times \mathbb{S}^1$  with applications to  $\mathbb{S}^2 \times \mathbb{R}$ , and very recently Manzano, Plehnert and the author [MPT13] constructed orientable and non-orientable even Euler characteristic embedded minimal surfaces in the quotient  $\mathbb{S}^2 \times \mathbb{S}^1$  and Martín, Mazzeo and Rogríquez [MMR14] constructed the first examples of complete, properly embedded minimal surfaces in  $\mathbb{H}^2 \times \mathbb{R}$  with finite total curvature and positive genus.

In this paper we are going to show a geometric relation between maximal surfaces in anti-De Sitter space-time  $\mathbf{H}_1^3$  and minimal immersions in  $\mathbb{H}^2 \times \mathbb{R}$ . It is well-known that the Gauss map of a spacelike maximal immersion in  $\mathbf{H}_1^3$  is always a minimal Lagrangian immersion in  $\mathbb{H}^2 \times \mathbb{H}^2$

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2000 *Mathematics Subject Classification.* Primary 53C42; Secondary 53C40.

*Key words and phrases.* Surfaces, minimal, complex surfaces.

Research partially supported by a MCyT-Feder research project MTM2011-22547, Junta Andalucía Grants P09-FQM-5088 and P09-FQM-4496 and Belgian Interuniversity Attraction Pole P07/18 (Dygest).

(see [Tor07] and also [CU07] where an analogous case for Lagrangian minimal immersions in  $S^2 \times S^2$  is studied). We are going to get new minimal immersions in  $\mathbb{H}^2 \times \mathbb{H}^2$  by pairing different components of the Gauss map of two suitable maximal immersions in anti-De Sitter space-time (see Theorem 1). This construction is in the same spirit as in [TU13]. From that, and under an appropriate choice of one element in the pair, we will establish a conformal correspondence between maximal immersions in anti-De Sitter space-time and minimal immersions in  $\mathbb{H}^2 \times \mathbb{R}$  (see Corollary 1). We will also show that this result admits a local converse, i.e. that roughly speaking every minimal surface in  $\mathbb{H}^2 \times \mathbb{R}$  is locally the Gauss map of a maximal immersion in anti-De Sitter space (see Theorem 2).

Finally, we will illustrate this geometric correspondence by showing new examples of maximal surfaces in anti-De Sitter space in Proposition 1 and computing their Gauss map (in the sense of Corollary 1). This will provide us with two 1-parameter families of minimal examples in  $\mathbb{H}^2 \times \mathbb{R}$ . We want to point out that, although the constructed maximal immersions in  $\mathbf{H}_1^3$  are non-complete (see Proposition 1), their corresponding minimal immersions in  $\mathbb{H}^2 \times \mathbb{R}$  induce complete metrics in the surface. Moreover, they are invariant by screw motions and they were first described in [ST04].

The structure of the paper is the following: Section 2 introduces both anti-De Sitter space-time  $\mathbf{H}_1^3$  and the Riemannian products  $\mathbb{H}^2 \times \mathbb{H}^2$  and  $\mathbb{H}^2 \times \mathbb{R}$ , where  $\mathbb{H}^2$  stands for the hyperbolic plane. In Section 3, we will briefly present some basic facts about maximal surfaces in anti-De Sitter space as well as some examples. Section 4 contains the main theorems that are illustrated in Section 5. Finally, Section 6 contains an analysis of the solutions to the sinh-Gordon equation that only depend on one variable.

## 2. PRELIMINARIES

**2.1. The hyperbolic plane and the product manifold  $\mathbb{H}^2 \times \mathbb{H}^2$ .** Let  $\mathbb{H}^2$  be the hyperbolic plane and  $\langle \cdot, \cdot \rangle$  its metric. Although for all computations we will use the hyperboloid model of  $\mathbb{H}^2$ , i.e.  $\mathbb{H}^2 = \{p \in \mathbb{R}^3 : \langle p, p \rangle = -1, p_1 > 0\}$ , where  $\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3$ , the Poincaré disc model is also considered in Figures 1 and 2.

We endow  $\mathbb{H}^2 \times \mathbb{H}^2$  with the product metric, also denoted by  $\langle \cdot, \cdot \rangle$ . So  $\mathbb{H}^2 \times \mathbb{H}^2$  is an Einstein manifold with constant scalar curvature  $-4$ .

We will consider  $\mathbb{H}^2 \times \mathbb{R}$  as the totally geodesic submanifold of  $\mathbb{H}^2 \times \mathbb{H}^2$  given by the image of the map  $i : \mathbb{H}^2 \times \mathbb{R} \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$  defined by  $i(p, t) = [p, (\sinh(t), 0, \cosh(t))]$ .

**2.2. The anti-De Sitter 3-space.** The anti-De Sitter 3-space, that it is usually denoted by  $\mathbf{H}_1^3$ , is a Lorentz manifold of dimension 3 and constant curvature  $-1$ . It is defined as a quadric in a vector space. More precisely, let  $\mathbb{R}_2^4$  be the euclidean 4-space endow with the metric  $\langle\langle u, v \rangle\rangle = u_1v_1 + u_2v_2 - u_3v_3 - u_4v_4$ . Then  $\mathbf{H}_1^3 = \{p \in \mathbb{R}_2^4 : \langle\langle p, p \rangle\rangle = -1\}$ . Moreover, the map  $\pi : \mathbf{H}_1^3 \subset \mathbb{R}_2^4 \equiv \mathbb{C}_1^2 \rightarrow \mathbb{H}^2(-2)$ , where  $\mathbb{H}^2(c)$  stands for the

hyperbolic plane of constant curvature  $c < 0$ , given by

$$\pi(z, w) = \left( z\bar{w}, \frac{1}{2}(|z|^2 + |w|^2) \right),$$

is a semi-Riemannian submersion with totally geodesic fibers generated by the unit temporal vector field  $\xi_{(z,w)} = (iz, iw)$ . The fiber of a point  $(z_0, w_0) \in \mathbf{H}_1^3$  is the circle  $(z_0 e^{it}, w_0 e^{it})$ .

**2.3. The Gauss map.** Let  $\phi : \Sigma \rightarrow \mathbf{R}_2^4$  be a spacelike immersion of an oriented surface  $\Sigma$ . Its Gauss map assigns to each point of the surface its oriented tangent plane. In this particular case, the image of the Gauss map is contained in  $G_*^+(2, 4)$ , the Grassmann manifold of oriented spacelike planes of  $\mathbf{R}_2^4$ . It is well-known that  $G_*^+(2, 4)$  is diffeomorphic to  $\mathbf{H}^2 \times \mathbf{H}^2$  (see, for example, [Pal91, Section 1]). To understand the construction in Section 4 a suitable identification between  $\mathbf{H}^2 \times \mathbf{H}^2$  and  $G_*^+(2, 4)$  is needed.

Let  $\Lambda^2 \mathbf{R}_2^4 = \text{span}\{v \wedge w : v, w \in \mathbf{R}_2^4\} \equiv \mathbf{R}^6$  be the linear space generated by the 2-vectors in  $\mathbf{R}_2^4$  endowed with the index 2 metric

$$g(v \wedge w, v' \wedge w') = \langle\langle v, w' \rangle\rangle \langle\langle w, v' \rangle\rangle - \langle\langle v, v' \rangle\rangle \langle\langle w, w' \rangle\rangle.$$

The star operator  $\star : \Lambda^2 \mathbf{R}_2^4 \rightarrow \Lambda^2 \mathbf{R}_2^4$  defined by  $\alpha \wedge (\star \beta) = g(\alpha, \beta) \Omega$  for all  $\alpha, \beta \in \Lambda^2 \mathbf{R}_2^4$ , where  $\Omega$  is the orientation form of  $\mathbf{R}_2^4$ , is a linear automorphism of  $\Lambda^2 \mathbf{R}_2^4$  with eigenvalues  $\pm 1$ . Consider  $\Lambda_{\pm}^2 \mathbf{R}_2^4$  the eigenspaces of  $\star$  associated to the eigenvalues  $\pm 1$ . Observe that we can decompose  $\Lambda^2 \mathbf{R}_2^4 = \Lambda_+^2 \mathbf{R}_2^4 \oplus \Lambda_-^2 \mathbf{R}_2^4$ . Let  $\{e_1, e_2, e_3, e_4\}$  be an oriented orthonormal frame of  $\mathbf{R}_2^4$ , i.e.  $|e_1|^2 = |e_2|^2 = -|e_3|^2 = -|e_4|^2 = 1$  and  $\langle\langle e_i, e_j \rangle\rangle = 0, i \neq j$ . The frame  $\{E_j^{\pm} : j = 1, 2, 3\}$  given by:

$$E_1^{\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_4 \wedge e_3), E_2^{\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2), E_3^{\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3),$$

is an orthonormal oriented reference in  $\Lambda_{\pm}^2 \mathbf{R}_2^4$ , i.e.  $g(E_i^{\pm}, E_j^{\pm}) = \epsilon_i \delta_{ij}$ , where  $\epsilon_1 = -1$  and  $\epsilon_2 = \epsilon_3 = 1$ . Hence each  $\Lambda_{\pm}^2 \mathbf{R}_2^4$  is isometric to the Lorentz-Minkowski 3-space. We denote by  $\mathbf{H}_{\pm}^2$  the hyperbolic plane in the 3-space  $\Lambda_{\pm}^2 \mathbf{R}_2^4$ .

Finally, if  $\{v, w\}$  is an oriented orthonormal frame of a plane  $P \in G_*^+(2, 4)$ , then the map  $G_*^+(2, 4) \rightarrow \mathbf{H}_+^2 \times \mathbf{H}_-^2$  given by

$$P \mapsto \frac{1}{\sqrt{2}}[v \wedge w + \star(v \wedge w), v \wedge w - \star(v \wedge w)]$$

is a diffeomorphism.

### 3. MAXIMAL SURFACES IN ANTI-DE SITTER SPACE

Let  $\phi : \Sigma \rightarrow \mathbf{H}_1^3$  be a spacelike maximal immersion, i.e. with zero mean curvature, of an oriented surface  $\Sigma$  and  $N$  a unit normal vector field to  $\phi$ . Given a conformal parameter  $z = x + iy$  on  $\Sigma$ , it is well-known (see for instance [Pal90]) that the 2-differential  $\Theta_{\phi}(z) = \theta(z) dz \otimes dz = \langle\langle \phi_z, N_z \rangle\rangle dz \otimes dz$  is holomorphic, where  $N$  is the (timelike) unit normal vector field to  $\phi$  such that  $\{\phi_x, \phi_y, \phi, N\}$  is a positively oriented frame in

$\mathbb{R}_2^4$  (we are using subscripts to indicate derivatives). The associated conformal factor  $e^{2v}$  satisfies  $v_{z\bar{z}} + e^{-2v} |\Theta_\phi|^2 - \frac{1}{4}e^{2v} = 0$ . Moreover, the Frenet equations of the immersion are given by

$$(3.1) \quad \phi_{zz} = 2v_z \phi_z + \theta N, \quad \phi_{z\bar{z}} = \frac{1}{2}e^{2v} \phi, \quad N_z = 2e^{-2v} \theta \phi_{\bar{z}}.$$

Conversely, we get the following result (see [Pal90, Proposition 2.1] and also [Per09, Lemma 3.3]):

*For any solution  $v : D \rightarrow \mathbb{R}$ ,  $D \subset \mathbb{C}$ , to the equation  $v_{z\bar{z}} - \frac{1}{2} \sinh(2v) = 0$  there exists a 1-parameter family  $\phi_t : \mathbb{C} \rightarrow \mathbf{H}_1^3$  of maximal immersions whose induced metric is  $e^{2v} |dz|^2$  and whose Hopf differential is  $\Theta_{\phi_t}(z) = \frac{i}{2}e^{it} dz \otimes dz$ .*

In this section we are going to present some examples of spacelike maximal surfaces in  $\mathbf{H}_1^3$  that will be useful in the sequel. The first simple example is the totally geodesic embedding of the hyperbolic plane  $\mathbb{H}^2$  into  $\mathbf{H}_1^3$  given by  $\mathbf{B} = \{(z, w) \in \mathbb{R}_2^4 \equiv \mathbb{C}^2 : \text{Im}(w) = 0\}$ , up to isometries of  $\mathbf{H}_1^3$ .

The second example, that will play an important role in the following section (see Corollary 1), is the so-called *hyperbolic cylinder*

$$\mathbf{C} = \{(z, w) \in \mathbb{R}_2^4 \equiv \mathbb{C}^2 : \text{Re}(z)^2 - \text{Re}(w)^2 = \text{Im}(z)^2 - \text{Im}(w)^2 = -\frac{1}{2}\}.$$

It is a complete spacelike maximal surface with vanishing Gauss curvature, constant principal curvatures  $\lambda_1 = -\lambda_2 = 1$  and the norm of the second fundamental form is  $|\sigma|^2 = 2$ . It was characterized by Ishihara [Ish88] as the only complete maximal surface in  $\mathbf{H}_1^3$ , up to rigid motions, with  $|\sigma|^2 = 2$ . We can parametrize the hyperbolic cylinder  $\mathbf{C}$  by

$$(3.2) \quad \psi_t(x, y) = \frac{1}{\sqrt{2}}(\sinh a_t(x, y), \sinh b_t(x, y), \cosh a_t(x, y), \cosh b_t(x, y)),$$

where

$$\begin{aligned} a_t(x, y) &= (x + y) \cos \frac{t}{2} + (x - y) \sin \frac{t}{2}, \\ b_t(x, y) &= (y - x) \cos \frac{t}{2} + (x + y) \sin \frac{t}{2}. \end{aligned}$$

Then  $\psi_t(\mathbb{R}^2) = \mathbf{C}$ ,  $z = x + iy$  is a conformal parameter with conformal factor  $e^{2u(x, y)}$  where  $u(x, y) = 0$ , and the associated Hopf differential is  $\Theta_{\psi_t} = \frac{i}{2}e^{it} dz \otimes dz$ .

Next we are going to show examples invariant under a 1-parameter group of isometries of  $\mathbf{H}_1^3$ . In that case, the equation for its conformal factor  $v_{z\bar{z}} - \frac{1}{2} \sinh(2v) = 0$  becomes an ordinary differential equation

$$(3.3) \quad v''(x) - 2 \sinh(2v(x)) = 0, \text{ with energy } E = \frac{1}{2}v'(x)^2 - \cosh(2v(x)).$$

It is possible to integrate explicitly the Frenet system (3.1) for some values of  $E$  obtaining the following result (see also Section 6 where we get all the solutions to the previous equation in terms of Jacobi elliptic functions).

**Proposition 1.** *Let  $v : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a solution to (3.3) with energy  $E$ . Then the map  $\phi_E : \Sigma_v = (I \times \mathbb{R}, e^{2v} g_0) \rightarrow \mathbf{H}_1^3$  given by:*

$$\begin{aligned} \phi_E(x, y) &= \frac{1}{\sqrt{2E}} \left( e^{v(x)} \cos(\sqrt{2E}y), -e^{v(x)} \sin(\sqrt{2E}y), \right. \\ &\quad \left. -\sqrt{2E + e^{2v(x)}} \cos(\sqrt{2E}G(x)), -\sqrt{2E + e^{2v(x)}} \sin(\sqrt{2E}G(x)) \right), E > 0, \\ \phi_E(x, y) &= \frac{1}{\sqrt{-2E}} \left( \sqrt{-2E - e^{2v(x)}} \sinh(\sqrt{-2E}G(x)), e^{v(x)} \sinh(\sqrt{-2E}y) \right. \\ &\quad \left. e^{v(x)} \cosh(\sqrt{-2E}y), \sqrt{-2E - e^{2v(x)}} \cosh(\sqrt{-2E}G(x)) \right), E < 0, \end{aligned}$$

is an isometric maximal immersion with associated Hopf differential  $\Theta(z) = \frac{1}{2} dz \otimes dz$ , where  $G(x) = \int_0^x \frac{dt}{2E + e^{2v(t)}}$  and  $g_0$  is the Euclidean metric in  $\mathbb{R}^2$ .

Moreover, all the surfaces  $\Sigma_v$ , except the one associated to the trivial solution  $v(x) = 0$  which is the hyperbolic cylinder, are not complete.

*Remark 1.* The obtained examples in Proposition 1 are invariant by the following 1-parameter group of isometries depending on the sign of  $E$ :

$$\begin{array}{cc} E > 0 & E < 0 \\ \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta & 0 \\ 0 & \sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Besides, when  $E < 0$  the immersion  $\phi_E$  is only defined for  $2E < -e^{2v(x)}$ . A deep analysis of the solutions (see Section 6) shows that there is always a non-empty interval  $I' \subseteq I$  where this happens, being  $I$  the maximal interval of definition of  $v$  (see also the proof of Proposition 1).

*Remark 2.* It is not surprising that the obtained examples are not complete. In [Per09], the author pointed out the difficulties of getting complete maximal immersions in anti-De Sitter space-time and he also provided with complete examples looking for radial solutions of the sinh-Gordon equation. Palmer in [Pal90, Theorem I] also provide complete minimal examples in  $\mathbf{H}_1^3$  in terms of holomorphic quadratic differentials.

*Proof.* Taking equation (3.3) into account, it is straightforward to check that  $\phi_E$  is a maximal isometric immersion. We will now show that the metric  $e^{2v} g_0$  is non-complete except for the trivial solution  $v = 0$ , by finding a divergent curve  $\gamma$  in  $\Sigma$  with finite length. Notice that we can consider the solutions to (3.3) given in Lemma 1 for  $a_0 = 0$ , which are always defined in an interval  $I = ]0, \ell[$ . Observe that if  $v$  is a solution then  $-v$  is also a solution (see Section 6) so we have to deal with both cases.

If  $E > -1$ , thanks to the symmetries of the solutions (see Remark 5.(1)),  $v$  and  $-v$  are symmetric so, considering the one given in Lemma 1 we have that  $e^{v(x)} \leq 1$  in  $]0, \ell/2[$ . Hence the curve  $\gamma : ]0, \ell/2[ \rightarrow \Sigma$ , given by  $\gamma(t) = \frac{\ell}{2} - t$ , diverges in  $\Sigma$  but has finite length.

If  $E = -1$  we have three different solutions: (1)  $v(x) = 0$ , which produces the hyperbolic cylinder which is complete; (2)  $v(x) = \log \tanh(x)$ ,

in this case  $\Sigma = (\mathbb{R}^+ \times \mathbb{R}, \tanh^2(x)g_0)$  and so the curve  $\gamma(t) = (a - t, 0)$ ,  $t \in ]0, a[$ ,  $a \in \mathbb{R}$  arbitrary, diverges in  $\Sigma$  but has finite length; and (3)  $v(x) = \log \coth(x)$ . In this case the immersion is only defined when  $\coth^2(x) < 2$  (see Remark 1), i.e.  $\Sigma = (]0, \operatorname{arccotanh}(\sqrt{2})[ \times \mathbb{R}, \coth^2(x)g_0)$ . Hence the curve  $\gamma(t) = (t, 0)$ ,  $t \in ]\frac{1}{2}, \operatorname{arccotanh}(\sqrt{2})[$  diverges in  $\Sigma$  but has finite length.

Finally, if  $E < -1$  we have two different types of solutions, namely  $v_1(x) = -\log(\frac{1}{\lambda} \operatorname{sn}_\mu(\lambda x))$  and  $v_2(x) = -v_1(x)$  (see Lemma 1). In the first case the immersion is only defined when  $2E + e^{2v_1(x)} < 0$ , i.e. for  $x \in J = ]c, \ell - c[$  where  $c = \frac{1}{\lambda} \operatorname{arcsn}_\mu(\frac{\lambda}{\sqrt{-2E}})$  (see Lemma 1 for the definition of  $\lambda$  and  $\mu$ ). But  $e^{v_1(x)} \leq \sqrt{-2E}$  for  $x \in ]c, \ell - c[$  and so  $\Sigma$  is also incomplete in this case.

In the second case,  $2E + e^{2v_2(x)} < 0$  so  $\Sigma = (]0, \ell[ \times \mathbb{R}, e^{2v_2(x)}g_0)$ , but  $e^{v_2(x)} \leq \frac{1}{\lambda}$  so the curve  $\gamma(t) = (\ell/2 - t, 0)$ ,  $t \in ]0, \ell/2[$ , diverges in  $\Sigma$  and has finite length.  $\square$

#### 4. THE GAUSS MAP OF A PAIR OF MAXIMAL SURFACES IN $\mathbf{H}_1^3$

Let  $\phi : \Sigma \rightarrow \mathbf{H}_1^3 \subset \mathbb{R}_2^4$  be a spacelike immersion of an oriented surface  $\Sigma$ . The Gauss map of  $\phi : \Sigma \rightarrow \mathbb{R}_2^4$  is the map  $\nu_\phi = (\nu_\phi^+, \nu_\phi^-) : \Sigma \rightarrow \mathbb{H}_+^2 \times \mathbb{H}_-^2$  defined by

$$\nu_\phi^\pm(p) = \frac{1}{\sqrt{2}}[e_1 \wedge e_2 \pm N(p) \wedge \phi(p)],$$

where  $\{e_1, e_2\}$  is an oriented orthonormal basis in  $T_p\Sigma$  and  $N$  is the unit (timelike) normal vector field to the immersion  $\phi$  such that  $\{e_1, e_2, \phi(p), N_p\}$  is oriented in  $\mathbb{R}_2^4$  (see Section 2.3).

If  $\phi$  is maximal then its Gauss map is a Lagrangian minimal immersion in  $\mathbb{H}^2 \times \mathbb{H}^2$  (see [Tor07]). For instance:

- The Gauss map of the totally geodesic embedding of the hyperbolic plane in  $\mathbf{H}_1^3$  given in Section 3 is the diagonal map  $\nu : \mathbb{H}^2 \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$ ,  $\nu(p) = (p, p)$ .
- The Gauss map of the hyperbolic cylinder is the product of two geodesics of  $\mathbb{H}^2$ .

**Theorem 1.** *Let  $\Sigma$  be a Riemann surface and  $\phi, \psi : \Sigma \rightarrow \mathbf{H}_1^3$  two conformal spacelike maximal immersions with the same Hopf differentials  $\Theta_\phi = \Theta_\psi$ . Then*

$$\nu_{\{\phi, \psi\}} : (\nu_\phi^+, \nu_\psi^-) : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$$

*is a conformal minimal immersion. Moreover, the induced metric by  $\nu_{\{\phi, \psi\}}$  is*

$$g = \frac{1}{2}[(2 + |\sigma_\phi|^2)g_\phi + (2 + |\sigma_\psi|^2)g_\psi],$$

*where  $g_\phi$  and  $g_\psi$  are the induced metrics on  $\Sigma$  by  $\phi$  and  $\psi$ , respectively. Here  $|\sigma_\phi|$  and  $|\sigma_\psi|$  are the lengths of the second fundamental forms of  $\phi$  and  $\psi$  in  $\mathbf{H}_1^3$ , computed with respect to  $g_\phi$  and  $g_\psi$ , respectively.*

*Remark 3.*

- (1) If  $\phi = \psi$ , then  $\nu_{\{\phi, \psi\}} = \nu_\phi$  is the Gauss map of  $\phi$ .

- (2) Given a maximal immersion  $\phi : \Sigma \rightarrow \mathbf{H}_1^3$ , its *polar* immersion (possibly branched) is  $N : \Sigma \rightarrow \mathbf{H}_1^3$ , where  $N$  is a unit normal vector field to  $\phi$ .  $N$  is also a maximal conformal immersion with the same Hopf differential as  $\phi$ . Nevertheless,  $\nu_{\{\phi, N\}}$  is congruent to  $\nu_\phi$ , the Gauss map of  $\phi$ .
- (3) Given  $A \in \mathcal{O}_2(4)$ , then it is easy to check that  $\nu_{\{A\phi, \psi\}}$  is congruent to  $\nu_{\{\phi, \psi\}}$ .

*Proof.* The immersion  $\phi : \Sigma \rightarrow \mathbb{R}_2^4$  has parallel mean curvature vector because it is contained in  $\mathbf{H}_1^3$  as a maximal surface. From [Pal91, Theorem 3.2] we deduce that each  $\nu_\phi^\pm$  is a harmonic map. Analogously  $\nu_\psi^\pm : \Sigma \rightarrow \mathbb{H}^2$  are also harmonic maps. Hence  $\nu_{\{\phi, \psi\}} = (\nu_\phi^+, \nu_\psi^-) : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$  is a harmonic map.

It remains to check that  $\nu$  is conformal (and so minimal). Let  $z = x + iy$  a conformal parameter over  $\Sigma$  and  $N_\phi, N_\psi$  the temporal unit normal vector field to  $\phi$  and  $\psi$  respectively such that  $\{\phi_x, \phi_y, \phi, N_\phi\}$  and  $\{\psi_x, \psi_y, \psi, N_\psi\}$  are oriented references in  $\mathbb{R}_2^4$ . Since  $\phi$  and  $\psi$  are conformal immersions the induced metrics by  $\phi$  and  $\psi$  in  $\Sigma$  are given by  $g_\phi = e^{2u} |dz|^2$  and  $g_\psi = e^{2w} |dz|^2$  for certain functions  $u$  and  $w$ . Moreover,  $\Theta_\phi = \Theta_\psi = \theta dz \otimes dz$  for some function  $\theta(z)$  by hypothesis.

We can express the component of the Gauss map  $\nu_{\{\phi, \psi\}}$  as

$$\begin{aligned} \nu_\phi^+(z) &= \frac{1}{\sqrt{2}} (-2ie^{-2u} \phi_z \wedge \phi_{\bar{z}} - \phi \wedge N_\phi), \\ \nu_\psi^-(z) &= \frac{1}{\sqrt{2}} (-2ie^{-2w} \psi_z \wedge \psi_{\bar{z}} + \psi \wedge N_\psi), \end{aligned}$$

Taking the Frenet equations (3.1) of  $\phi$  and  $\psi$  and Section 2.3 into account, we easily get that

$$\begin{aligned} (\nu_\phi^+)_z &= \frac{1}{2} e^u (-i + 2\theta e^{-2u}) E_2^+(z) + \frac{i}{2} e^u (i + 2\theta e^{-2u}) E_3^+(z), \\ (\nu_\psi^-)_z &= \frac{1}{2} e^w (-i - 2\theta e^{-2w}) E_2^-(z) + \frac{i}{2} e^w (-i + 2\theta e^{-2w}) E_3^-(z). \end{aligned}$$

Then we deduce from the previous equations that:

$$\begin{aligned} \langle (\nu_\phi^+)_z, (\nu_\phi^+)_z \rangle &= -2i\theta, & \langle (\nu_\psi^-)_z, (\nu_\psi^-)_z \rangle &= 2i\theta, \\ |\nu_\phi^+|^2 &= \frac{1}{2} (e^{2u} + 4e^{-2u} |\theta|^2), & |\nu_\psi^-|^2 &= \frac{1}{2} (e^{2w} + 4e^{-2w} |\theta|^2). \end{aligned}$$

Finally,  $\langle \nu_z, \nu_z \rangle = 0$  and so  $\nu = \nu_{\{\phi, \psi\}}$  is a conformal map. Moreover, from  $8|\theta|^2 = e^{4u} |\sigma_\phi|^2 = e^{4w} |\sigma_\psi|^2$  and

$$|\nu_z|^2 = \frac{1}{2} (e^{2u} + e^{2w} + 4|\theta|^2 (e^{-2u} + e^{-2w})),$$

we get the expression of the induced metric on  $\Sigma$  by  $\nu$ .  $\square$

Now, let  $\phi : \Sigma \rightarrow \mathbf{H}_1^3$  be a conformal maximal immersion. There is no loss of generality in assuming that locally the Hopf differential  $\Theta(z) = \frac{i}{2} e^{it} dz \otimes dz$  (observe that either  $\Theta = 0$  and so the immersion is totally geodesic or the zeroes of  $\Theta$  are isolated and we can locally normalize  $\Theta$  away from the zeroes). Then  $\phi$  and  $\psi_t$ , the immersion of the hyperbolic cylinder given in Section 3, are two conformal maximal immersions with

the same Hopf differentials. Then, thanks to the previous theorem,  $\hat{v}_\phi = v_{\{\phi, \psi_t\}} : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$  is a minimal immersion that we call the *modified Gauss map* of  $\phi$ . Now, the Gauss map of the hyperbolic cylinder  $\psi_t$  is the product of two geodesics in  $\mathbb{H}^2 \times \mathbb{H}^2$  so its second component can be viewed as a map from  $\Sigma$  to  $\mathbb{R}$ . Hence the *modified Gauss map of a maximal immersion*  $\phi : \Sigma \rightarrow \mathbf{H}_1^3$  is a conformal minimal immersion  $\hat{v}_\phi : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}$ . We get the following result:

**Corollary 1.** *Let  $u : D \subseteq \mathbb{C} \rightarrow \mathbb{R}$  be a solution of  $u_{z\bar{z}} - \frac{1}{2} \sinh(2u) = 0$ . Then the 1-parameter family of minimal immersions  $\Phi_t : (D, 4 \cosh^2 u |dz|^2) \rightarrow \mathbb{H}^2 \times \mathbb{R}$  with Hopf differential  $\Theta = e^{it} dz \otimes dz$  (see [HST08, Corollary 10]) associated to  $u$  is given by:*

$$\Phi_t(z) = (v_{\phi_t}^+(z), 2 \operatorname{Im}(ze^{it/2})),$$

where  $\phi_t : (D, e^{2u} |dz|^2) \rightarrow \mathbf{H}_1^3$  is the 1-parameter family of immersions associated to  $u$  with Hopf differential  $\Theta_{\phi_t} = \frac{i}{2} e^{it} dz \otimes dz$ .

*Proof.* Let  $\psi_t = (\mathbb{C}, |dz|^2) \rightarrow \mathbf{H}_1^3$  the immersion of the hyperbolic cylinder given in equation (3.2). Then  $\phi_t, \psi_t : D \subseteq \mathbb{C} \rightarrow \mathbf{H}_1^3$  are conformal maximal immersions with the same Hopf differentials. Hence, applying the previous theorem we get that  $v_t = v_{\{\phi_t, \psi_t\}}$  is a conformal minimal immersion in  $\mathbb{H}^2 \times \mathbb{H}^2$  with induced metric  $4 \cosh^2 u |dz|^2$ .

Furthermore, a straightforward computation shows that

$$v_\psi^-(z) = \cosh[2 \operatorname{Im}(ze^{it/2})] E_1^- + \sinh[2 \operatorname{Im}(ze^{it/2})] E_3^-,$$

where  $\{E_1^-, E_2^-, E_3^-\}$  is the orthonormal reference in  $\Lambda_-^2 \mathbb{R}_2^4$  associated with the canonical base of  $\mathbb{R}_2^4$  (see Section 2.3). Hence, as we have mentioned above,  $v_\psi^-(D)$  is contained in a geodesic of  $\mathbb{H}^2$ . Considering  $\mathbb{R}$  embedded in  $\mathbb{H}^2$  as such geodesic we get the result. Finally, it is easy to check that  $\Theta_{v_t} = -2i\Theta_{\phi_t} = \Theta$ .  $\square$

The next result is a local converse of Theorem 1 in the special case of surfaces immersed in  $\mathbb{H}^2 \times \mathbb{R}$ .

**Theorem 2.** *Let  $\phi : \Sigma \rightarrow \mathbb{H}^2 \times \mathbb{R}$  an isometric minimal immersion of a simply connected Riemann surface  $\Sigma$  satisfying  $v^2 < 1$ , where  $v = \langle N, \partial_t \rangle$ . Then, there exists a conformal maximal immersion  $\psi : \Sigma \rightarrow \mathbf{H}_1^3$  such that  $\phi = \hat{v}_\psi$ , up to an ambient isometry.*

*Proof.* Let  $w$  a conformal parameter over  $\Sigma$ . Then  $Y(w) = \langle \phi_w, \partial_t \rangle dw$  is a holomorphic 1-form without zeroes (note that  $|Y|^2 = \frac{1}{4}(1 - v^2) > 0$  by assumption). Then we can always find another conformal parameter  $z$  such that  $Y = dz$ . The conformal factor induced by  $\phi$  in this new parameter is  $4 \cosh^2 u$ , where  $u = \tanh(v)$  satisfies  $u_{z\bar{z}} - \frac{1}{2} \sinh(2u) = 0$ . Moreover, the fundamental data of the immersion  $\phi$  can be expressed in terms of  $u$  (see [FM10, Theorem 2.3]).

Let  $\psi : \Sigma \rightarrow \mathbf{H}_1^3$  be the maximal conformal immersion associated to  $u$  with Hopf differential  $\Theta_\psi(z) = \frac{i}{2} dz \otimes dz$  (see Section 3). Then,  $\hat{v}_\psi : \Sigma \rightarrow$



$\mathbb{H}^2 \times \mathbb{R}$  is a minimal immersion with the same fundamental data as  $\phi$  and so both immersions differ in an ambient isometry.  $\square$

*Remark 4.* It is possible to get a similar result for minimal immersion of  $\mathbb{H}^2 \times \mathbb{H}^2$  without complex points as in [TU13, Theorem 3], that is, every minimal immersion in  $\mathbb{H}^2 \times \mathbb{H}^2$  without complex points is locally congruent to the Gauss map of the pair of two maximal immersion in the anti-De Sitter space-time.

## 5. EXAMPLES

In this section we are going to use Corollary 1 to compute the minimal immersions associated to the maximal immersions in  $\mathbf{H}_1^3$  given by Proposition 1. As we shall see, the obtained examples are invariant by 1-parameter groups of isometries of  $\mathbb{H}^2 \times \mathbb{R}$ , namely, *elliptic* and *hyperbolic* screw motions (see figures 1 and 2). Moreover, although the considered maximal immersions in  $\mathbf{H}_1^3$  are not complete (see Remark 1), their Gauss maps, in the sense of Corollary 1, are complete immersions.

Let  $v : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a solution of  $v''(x) - 2 \sinh(2v) = 0$  with energy  $E$  (cf. equation (3.3)). Then, the map  $\Phi_E : \Sigma = (I \times \mathbb{R}, 4 \cosh^2(v)g_0) \rightarrow \mathbb{H}^2 \times \mathbb{R}$  given by:

$$\begin{aligned} \Phi_E(x, y) &= \frac{1}{\sqrt{2E}} \left( v'(x), \frac{\sqrt{2E}e^{-v(x)} \cos \sqrt{2E}(y-G(x)) - v'(x)e^{v(x)} \sin \sqrt{2E}(y-G(x))}{\sqrt{2E+e^{2v(x)}}}, \right. \\ &\quad \left. \frac{\sqrt{2E}e^{-v(x)} \sin \sqrt{2E}(y-G(x)) + v'(x)e^{v(x)} \cos \sqrt{2E}(y-G(x))}{\sqrt{2E+e^{2v(x)}}}, 2\sqrt{E}(y-x) \right), \quad E > 0, \\ \Phi_E(x, y) &= \frac{1}{\sqrt{-2E}} \left( \frac{\sqrt{-2E}e^{-v(x)} \cosh \sqrt{-2E}(y-G(x)) - v'(x)e^{v(x)} \sinh \sqrt{-2E}(y-G(x))}{\sqrt{-2E-e^{2v(x)}}}, v'(x), \right. \\ &\quad \left. \frac{-\sqrt{-2E}e^{-v(x)} \sinh \sqrt{-2E}(y-G(x)) + v'(x)e^{v(x)} \cosh \sqrt{-2E}(y-G(x))}{\sqrt{-2E-e^{2v(x)}}}, 2\sqrt{-E}(y-x) \right), \quad E < 0, \end{aligned}$$

is an isometric minimal immersion with associated Hopf differential  $\Theta = -idz \otimes dz$ , where  $G(x) = \int_0^x \frac{dt}{2E+e^{2v(t)}}$  and  $g_0$  stands for the Euclidean metric in  $\mathbb{R}^2$ .

This can be checked directly from the definition or, taking into account Corollary 1, computing the first component of the Gauss map of the maximal immersion  $\phi_E$  given in Proposition 1.

These examples are invariant by the 1-parameter group of isometries  $A_\theta \times \tau_\theta$  of  $\mathbb{H}^2 \times \mathbb{R}$ , where  $A_\theta$  is an isometry of  $\mathbb{H}^2$  given by:

$$A_\theta = \begin{matrix} E > 0 & E < 0 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} & \begin{pmatrix} \cosh \theta & 0 & \sinh \theta \\ 0 & 1 & 0 \\ \sinh \theta & 0 & \cosh \theta \end{pmatrix} \end{matrix}$$

and  $\tau_\theta : \mathbb{R} \rightarrow \mathbb{R}$  is given by  $\tau_\theta(t) = t + \frac{\theta}{\sqrt{|E|}}$ .

The complete classification of constant mean curvature surfaces (in particular the minimal ones) invariant by a 1-parameter group of  $\mathbb{H}^2 \times \mathbb{R}$  can be found in [Onn08] and the references therein.

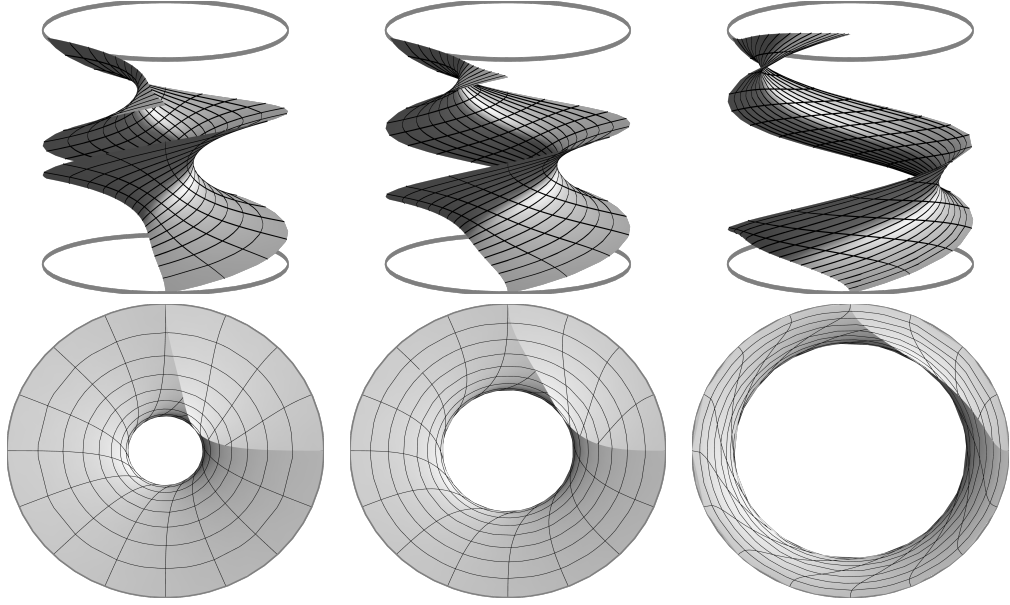


FIGURE 1. From left to right, typical solutions for positive energy  $E = 4, 1, 0.1$  in  $\mathbb{H}^2 \times \mathbb{R}$  being  $\mathbb{H}^2$  the disc model. Below each surface the top view has been drawn. The boundary of  $\mathbb{H}^2$  is drawn to help the visualization.

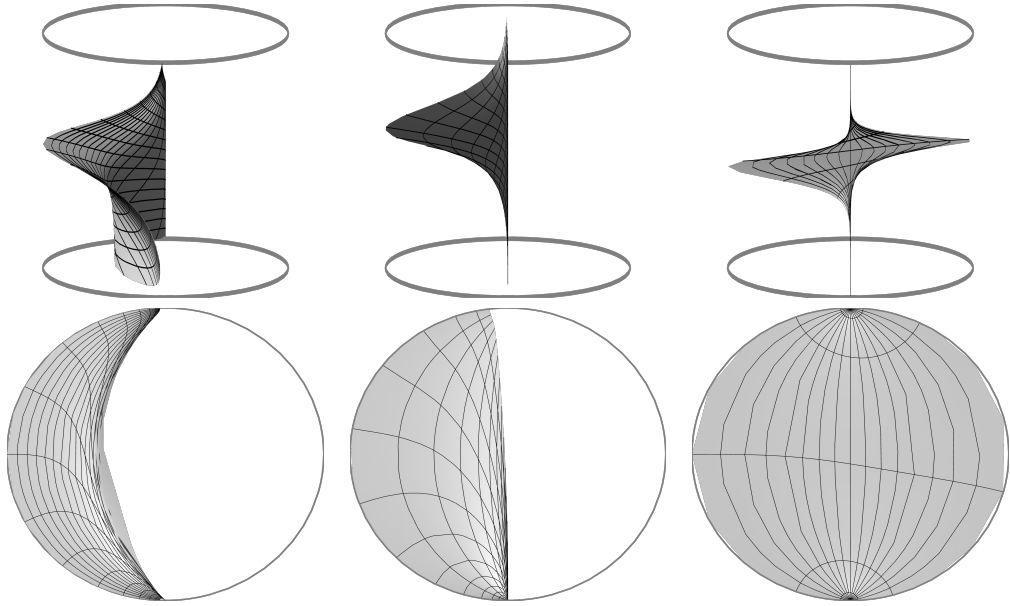


FIGURE 2. From left to right, typical solutions for negative energy  $E = -0.5, -1, -6$  in  $\mathbb{H}^2 \times \mathbb{R}$  being  $\mathbb{H}^2$  the disc model. Below each surface the top view has been drawn.

## 6. APPENDIX

In this section we will exhibit explicit solutions for the equation  $\Delta v - 2 \sinh(2v) = 0$ . We will restrict ourselves to the simplest case, that is, when the function only depends on one variable, i.e.  $v = v(x)$ . In that case it is easy to find a first integral of the equation, namely the energy  $E = (v')^2/2 - \cosh(2v)$  is constant for every solution  $v$  (cf. (3.3)). Moreover, if  $v$  is a solution then  $u(x) = -v(x)$  and  $w(x) = v(-x)$  are also solutions with the same energy of  $v$ . Hence, we only need to consider initial conditions  $v(0) = v_0 \geq 0$  and  $v'(0) = \sqrt{2(E + \cosh(2v_0))} \geq 0$  (note that  $E + \cosh(2v(x)) \geq 0$  by the definition of  $E$ ). Thus we are interested in solving the following initial value problem:

$$(6.1) \quad \begin{aligned} v''(x) - 2 \sinh(2v(x)) &= 0, \\ v(0) = v_0 \geq 0, \quad v'(0) &= \sqrt{2(E + \cosh(2v_0))}. \end{aligned}$$

It is possible to obtain all the solutions to that problem in terms of Elliptic Jacobi functions (see for instance [BF71] for further details). Let

$$F(\varphi, \mu) = \int_0^\varphi \frac{d\theta}{\sqrt{1 - \mu \sin^2 \theta}}, \quad 0 \leq \mu \leq 1,$$

be the *elliptic integral of the first kind with modulus  $\mu$* . Then denoting the inverse of  $\varphi \mapsto F(\varphi, \mu)$  by  $\varphi = \operatorname{am}_\mu(x)$ , the elementary Jacobi elliptic functions are given by:

$$\begin{aligned} \operatorname{sn}_\mu(x) &= \sin \operatorname{am}_\mu(x), & \operatorname{dn}_\mu(x) &= \sqrt{1 - \mu \sin^2 \operatorname{am}_\mu(x)} \\ \operatorname{cn}_\mu(x) &= \cos \operatorname{am}_\mu(x), & \operatorname{tn}_\mu(x) &= \operatorname{sn}_\mu(x) / \operatorname{cn}_\mu(x) \end{aligned}$$

The basic properties of these functions are:

$$\begin{aligned} \operatorname{sn}_\mu(x)^2 + \operatorname{cn}_\mu(x)^2 &= 1, & \mu \operatorname{sn}_\mu(x)^2 + \operatorname{dn}_\mu(x)^2 &= 1 \\ \operatorname{sn}_\mu(x + 2K(\mu)) &= -\operatorname{sn}_\mu(x), & \operatorname{cn}_\mu(x + 2K(\mu)) &= -\operatorname{cn}_\mu(x), \\ \operatorname{dn}_\mu(x + 2K(\mu)) &= \operatorname{dn}_\mu(x), & \operatorname{tn}_\mu(x + 2K(\mu)) &= \operatorname{tn}_\mu(x), \end{aligned}$$

where  $K(\mu) = F(\frac{\pi}{2}, \mu)$  is the complete elliptic integral of the first kind. Moreover, the derivatives of the Jacobi elliptic functions are:

$$\begin{aligned} \operatorname{sn}'_\mu(x) &= \operatorname{cn}_\mu(x) \operatorname{dn}_\mu(x), & \operatorname{cn}'_\mu(x) &= -\operatorname{sn}_\mu(x) \operatorname{dn}_\mu(x), \\ \operatorname{am}'_\mu(x) &= \operatorname{dn}_\mu(x), & \operatorname{dn}'_\mu(x) &= -\mu \operatorname{sn}_\mu(x) \operatorname{cn}_\mu(x). \end{aligned}$$

**Lemma 1.** *The solution  $v : I \rightarrow \mathbb{R}$  of the initial value problem (6.1) and its maximal definition interval  $I$  are given, in terms of the energy  $E$ , by:*

$$\begin{aligned}
[E > 1] \quad & v(x) = \log(\lambda \operatorname{tn}_\mu(\lambda^{-1}x + a_0)), & \mu &= 1 - \lambda^4, \\
& I = ] - a_0, \lambda K(\mu) - a_0[, & a_0 &= \operatorname{arctn}_\mu(\lambda^{-1}e^{v_0}), \\
[|E| \leq 1] \quad & v(x) = \log(\operatorname{tn}_\mu(x + a_0) \operatorname{dn}_\mu(x + a_0)), & \mu &= \frac{1-E}{2}, \\
& I = ] - a_0, K(\mu) - a_0[, & a_0 &= \frac{1}{2} \operatorname{arccn}_\mu(\tanh(v_0)), \\
[E < -1] \quad & v(x) = -\log(\lambda^{-1} \operatorname{sn}_\mu(\lambda x + a_0)), & \mu &= \lambda^{-4}, \\
& I = ] - a_0, 2\lambda^{-1}K(\mu) - a_0[, & a_0 &= \operatorname{arcsn}_\mu(\lambda e^{-v_0})
\end{aligned}$$

where  $\lambda^2 = |E - \sqrt{E^2 - 1}|$  for  $|E| > 1$ .

*Proof.* It is a direct computation taking into account the aforementioned properties of the Jacobi elliptic functions.  $\square$

*Remark 5.* In the special cases  $E = 1$  and  $E = -1$  we get solutions in terms of elementary functions, namely,  $v_1(x) = \log \tan(x)$ ,  $v_{-1}(x) = \log \cotanh(x)$  as well as the constant solution  $v(x) = 0$  (also with  $E = -1$ ).

On the one hand, the solutions of the sinh-Gordon equation with energy  $E > -1$  are symmetric with respect to the middle point of the maximal interval of definition. On the other hand, the solutions  $v$  with energy  $E < -1$  never vanish and are symmetric with respect to the vertical line passing through the middle point of the maximal interval of definition.

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